Exercise 11

Solve the initial-value problem.

$$y'' + 6y' = 0, \quad y(1) = 3, \quad y'(1) = 12$$

Solution

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y = e^{rx}$.

$$y = e^{rx} \quad \to \quad y' = re^{rx} \quad \to \quad y'' = r^2 e^{rx}$$

Substitute these formulas into the ODE.

$$r^2 e^{rx} + 6(re^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 6r = 0$$

Solve for r.

$$r(r+6) = 0$$

$$r = \{-6, 0\}$$

Two solutions to the ODE are e^{-6x} and $e^0 = 1$. According to the principle of superposition, the general solution is a linear combination of these two.

$$y(x) = C_1 e^{-6x} + C_2$$

Differentiate it with respect to x.

$$y'(x) = -6C_1 e^{-6x}$$

Apply the initial conditions to determine C_1 and C_2.

$$y(1) = C_1 e^{-6} + C_2 = 3$$

$$y'(1) = -6C_1 e^{-6} = 12$$

Solve the system.

$$C_1 = -2e^6 \qquad C_2 = 5$$

Therefore,

$$\begin{aligned} y(x) &= (-2e^6)e^{-6x} + 5 \\ &= -2e^{6(1-x)} + 5. \end{aligned}$$

Below is a plot of the solution versus x.